Numerical Integration: Trapezoidal Rule vs Gaussian Quadrature

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1 Introduction

We seek to evaluate the integral

$$I = \int_0^2 \frac{1}{\sqrt{1 + \sin^2(x)}} \, dx \tag{1}$$

This integral does not have an elementary closed-form solution. It is an *elliptic integral*[1]. Therefore, we must use numerical methods. We compare the **Trapezoidal rule** and **Gaussian quadrature** in terms of accuracy and computational efficiency.

2 Numerical Methods

2.1 Trapezoidal Rule

The composite trapezoidal rule approximates the integral as:

$$I \approx \frac{h}{2} \left[f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b) \right] \tag{2}$$

where $h = (b - a)/n$ is the step size.

2.2 Gaussian Quadrature

Gaussian quadrature uses optimally chosen nodes and weights:

$$I \approx \sum_{i=1}^{n} w_i f(x_i) \tag{3}$$

For smooth functions, Gaussian quadrature achieves much higher accuracy with fewer function evaluations, see e.g., Golub and Welsch (1969) and Cenanovic, Jansson, and Jonsson (2026).

[1] https://en.wikipedia.org/wiki/Elliptic_integral

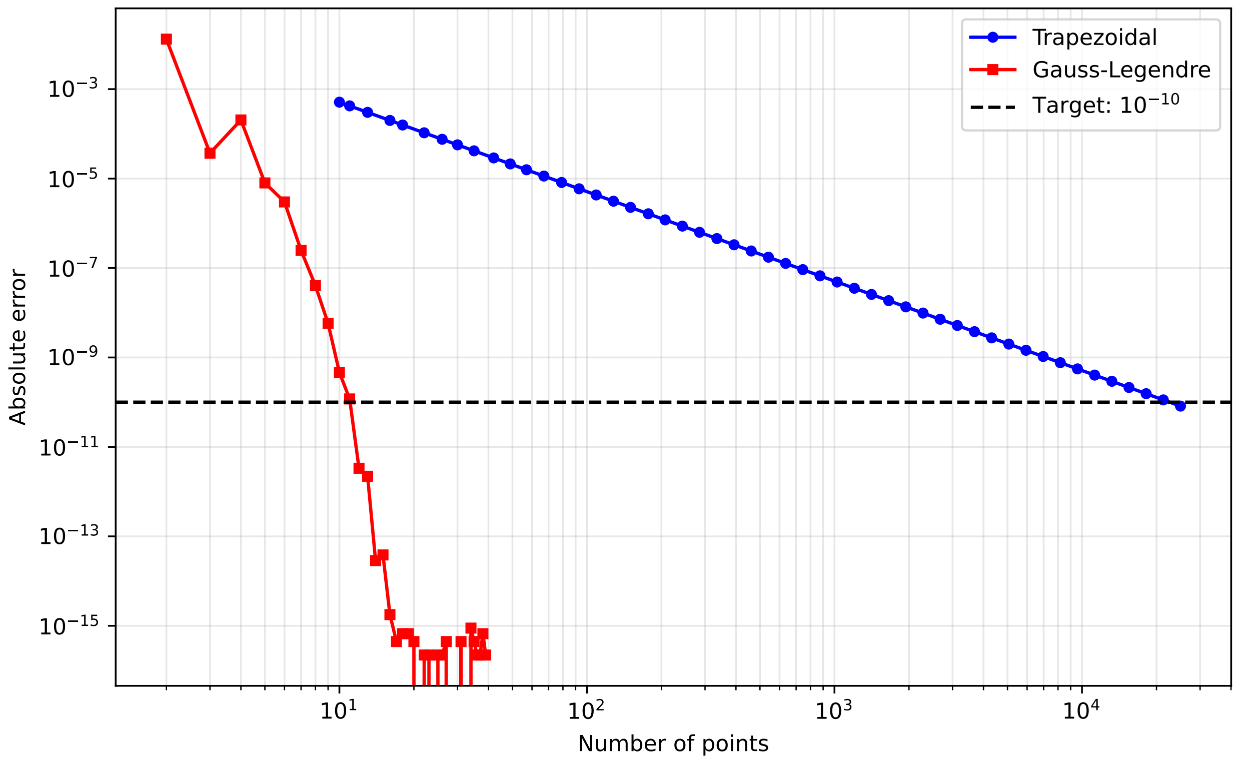

Figure 1: Convergence comparison: error vs number of points for both methods.

Table 1: Points required and computation time to achieve error $< 10^{-10}$.

Method	Points Required	Time (s)
Trapezoidal	22602	209.3
Gauss-Legendre	12	202.7

3 Conclusion

Gaussian quadrature dramatically outperforms the trapezoidal rule for this smooth integrand. To achieve an error below 10^{-10}, trapezoidal rule requires thousands of points whereas Gaussian rule requires only 12 points. See Figure 1 for the convergence comparison and Table 1 for the numerical results.

This demonstrates the power of Gaussian quadrature for smooth functions, where it achieves *exponential* convergence compared to the *algebraic* ($O(h^2)$) convergence of the trapezoidal rule.

Cenanovic, Mirza, Johan Jansson, and Carl-Johan Jonsson. 2026. "Numerical Integration." 2026. https://python.ju.se/AppliedMaths/Numerical_Integration.html.

Golub, Gene H., and John H. Welsch. 1969. "Calculation of Gauss Quadrature Rules." *Mathematics of Computation* 23 (106): 221–30.